The Solution Space of Polynomial Equations with Real Roots and some of its Implications

Kenan Kalajdzic <kenan@unix.ba>

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1 Introduction

It is known from mathematics and physics that a state of a system can be represented by a single unique point in a phase space. For each such point in the phase space, there is a corresponding radius vector. This means that, if we know the magnitude of the radius vector and the angles it forms with each of the coordinate axes of the phase space, then we know the state of the system.

By using an analogy to the phase space, we can define a space in which every equation of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \tag{1}$$

whose all solutions are real, will be represented by a single point.

2 Basic definitions

Let us assume that equation (1) of n-th order is given and let us also assume this equation has n real solutions $x_{10}, x_{20}, \ldots, x_{n0}$. In an n-dimensional Euclidean space with coordinates $x_1, x_2, \ldots, x_n$ we could assign one of the solutions of the equation (1) to each of the coordinate axes $x_1, x_2, \ldots, x_n$, so that the solution x_{i0} is assigned to axis x_i where $i = 1, 2, \ldots, n$. It is obvious that in this space, the equation (1) will be defined by a single point $(x_{10}, x_{20}, \ldots, x_{n0})$ with a corresponding radius vector $\mathbf{X_0} = \{x_{10}, x_{20}, \ldots, x_{n0}\}$. A space, defined in this manner, is called the *solution space of the polynomial equation* (1), which we will denote by $\mathbb{S}^n$.

The magnitude of the vector $\mathbf{X_0}$ is given by the following expression

$$|\mathbf{X_0}| = \sqrt{x_{10}^2 + x_{20}^2 + \cdots + x_{n0}^2}$$

and is called the *solution radius of the polynomial equation* (1).

If the value of the solution radius is known, determining the components of the vector $\mathbf{X_0}$, and thus solving the equation (1), requires finding the angles that $\mathbf{X_0}$ forms with the coordinate axes $x_1, x_2, \ldots, x_n$ of the solution space $\mathbb{S}^n$.

If by φ_i we denote the angle, which is formed by the vector $\mathbf{X_0}$ and axis x_i of $\mathbb{S}^n$, then we can write the solutions of the equation (1) in the form

$$x_{i0} = |\mathbf{X_0}| \cos\varphi_i \qquad i = 1, 2, \ldots, n$$

$\varphi_1, \varphi_2, \ldots, \varphi_n$ are called the *solution angles of the polynomial equation* (1).

Since $|\cos\varphi_i| \leq 1$ for $i = 1, 2, \ldots, n$, it is obvious that all solutions of equation (1) lie in the closed interval $[-|\mathbf{X_0}|, +|\mathbf{X_0}|]$.

3 Theorem of n real roots

Theorem. *Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial function, whose coefficients $a_n, a_{n-1}, \ldots, a_1$, a_0 are real numbers with $a_n \neq 0$. Let us assume that all zeros of $p(x)$ are real numbers. Then all n roots of the equation $p(x) = 0$ lie in the closed interval $[-R, +R]$, where R is given by the following expression*

$$R = \sqrt{\left(\frac{a_{n-1}}{a_n}\right)^2 - \frac{2a_{n-2}}{a_n}} \qquad R \in \mathbb{R}, R \geq 0 \tag{2}$$

Proof. Let $x_{10}, x_{20}, \ldots, x_{n0}$ be the zeros of the function $p(x)$. Then, based on the Viète's formulas, we can write:

$$-\frac{a_{n-1}}{a_n} = \sum_{i=1}^{n} x_{i0} \tag{3}$$

$$\frac{a_{n-2}}{a_n} = \sum_{\substack{i,j=1 \\ i<j}}^{n} x_{i0}x_{j0} \tag{4}$$

By squaring equation (3), we get

$$\left(\frac{a_{n-1}}{a_n}\right)^2 = \sum_{i=1}^{n} x_{i0}^2 + 2\sum_{\substack{i,j=1 \\ i<j}}^{n} x_{i0}x_{j0} \tag{5}$$

The first term on the right-hand side of equation (5) is the square of the solution radius of the equation $p(x) = 0$ in the n-dimensional solution space $\mathbb{S}^n$. We will denote this solution radius with R. The second term on the right-hand side of (5) is obviously equal to the right-hand side of (4), so, after substituting (4) into (5), we can write

$$\left(\frac{a_{n-1}}{a_n}\right)^2 = R^2 + 2\frac{a_{n-2}}{a_n}$$

By solving this equation for R, we get the expression given in (2), where $a_{n-1}^2 \geq 2a_n a_{n-2}$. Since R is the solution radius of $p(x) = 0$, we conclude that all zeroes of $p(x)$ lie in the closed interval $[-R, +R]$ with R given by equation (2). This concludes the proof. $\square$

One of the assumptions of our theorem is that all zeroes of the polynomial function $p(x)$ are real numbers. Since the solution radius R is by definition a non-negative real number, we concluded that the expression under the square root on the right-hand side of (2) must also be non-negative. Therefore, the necessary condition for all roots of the polynomial equation (1) to be real is that a_{n-1}^2 be greater or equal to $2a_n a_{n-2}$.

4 Case study: The quadratic equation

In this section we follow the process of solving the quadratic equation in the context of its solution space. Although solving the quadratic equation may not sound very appealing, the fact that it is performed within a two-dimensional solution space gives us the possibility to work with fairly simple trigonometric transformations, which are also suitable for visual representation.

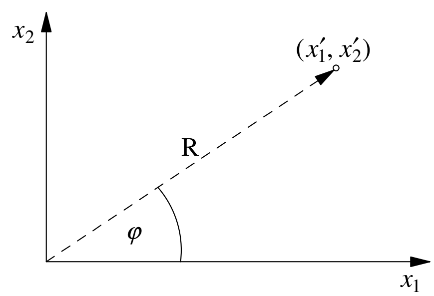

Figure 1: Solution of the quadratic equation in a two-dimensional solution space

Figure 1 illustrates a two-dimensional solution space of the quadratic equation

$$ax^2 + bx + c = 0. \tag{6}$$

The solution of this equation is represented by the point (x_1', x_2'), and the corresponding Viéte's formulas take the following form:

$$-\frac{b}{a} = x_1' + x_2' \tag{7}$$

$$\frac{c}{a} = x_1' x_2' \tag{8}$$

If we represent the solution of equation (6) in polar coordinates, we can write

$$x_1' = R\cos\varphi \tag{9}$$

$$x_2' = R\sin\varphi \tag{10}$$

Adding the equations (9) and (10) together gives

$$x_1' + x_2' = R(\sin\varphi + \cos\varphi) \tag{11}$$

After substituting (7) into (11) and squaring the resulting equation we get

$$\begin{aligned}\left(-\frac{b}{a}\right)^2 &= R^2(\sin\varphi + \cos\varphi)^2 = \\ &= R^2(\sin^2\varphi + 2\sin\varphi\cos\varphi + \cos^2\varphi) = R^2(1+\sin 2\varphi)\end{aligned}$$

and thus

$$\left(\frac{b}{a}\right)^2 = R^2 + R^2\sin 2\varphi \tag{12}$$

Now we multiply equations (9) and (10), so we have

$$\begin{aligned}x_1' x_2' &= R^2\cos\varphi\sin\varphi = \\ &= \frac{1}{2}R^2\ 2\sin\varphi\cos\varphi = \frac{1}{2}R^2\sin 2\varphi\end{aligned}$$

which, together with (8), gives

$$\frac{2c}{a} = R^2\sin 2\varphi \tag{13}$$

After substituting (13) into (12) and solving for R, we finally get

$$R = \sqrt{\left(\frac{b}{a}\right)^2 - \frac{2c}{a}} \tag{14}$$

We can see that equation (14) is the specific form of (2) for the polynomial equation given in (6). In this particular case, since we only have to deal with two variables in the polar coordinate system, and we already know the radius R of the position vector corresponding to the point (x_1', x_2'), we can easily calculate the second unknown, which is the angle φ. To do so, we substitute R from (14) back into (13):

$$\frac{2c}{a} = \left(\left(\frac{b}{a}\right)^2 - \frac{2c}{a}\right)\sin 2\varphi$$

Solving this equation for φ gives

$$\varphi = \frac{1}{2}\arcsin\left(\frac{2ac}{b^2 - 2ac}\right) \tag{15}$$

Equations (9) and (10) express the solution of the quadratic equation (6) in polar coordinates of the two-dimensional solution space. If we substitute (14) and (15) into (9) and (10), we get:

$$x_1' = \sqrt{\left(\frac{b}{a}\right)^2 - \frac{2c}{a}}\ \ \cos\left(\frac{1}{2}\arcsin\left(\frac{2ac}{b^2 - 2ac}\right)\right) \tag{16}$$

$$x_2' = \sqrt{\left(\frac{b}{a}\right)^2 - \frac{2c}{a}}\ \ \sin\left(\frac{1}{2}\arcsin\left(\frac{2ac}{b^2 - 2ac}\right)\right) \tag{17}$$

Equations (16) and (17) represent a novel form of the solutions of the quadratic equation (6), expressed in terms of its coefficients a, b and c, yet different from the commonly known form

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

5 Conclusion

Introducing the concept of a *solution space* of the polynomial equation (1) in Section 2 helped us formulate and prove the theorem of n real roots in Section 3. This theorem provides a simple way to determine the boundaries of the interval which contains all the roots of equation (1). For $n > 4$, the roots of this polynomial equation can only be found numerically, in which case it is useful to know where all these solutions reside.

One of the limitations of the presented approach is the fact that it requires all the roots of our polynomial equation (1) to be real. This also means that equations (16) and (17) are only valid if the quadratic equation (6) has real solutions.

How the paradigm of the solution space would be affected by polynomial equations with complex roots, remains a question for further research.